

Decomposition of Spectra from the Drum

With Applications to the Chesapeake Bay

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Outline

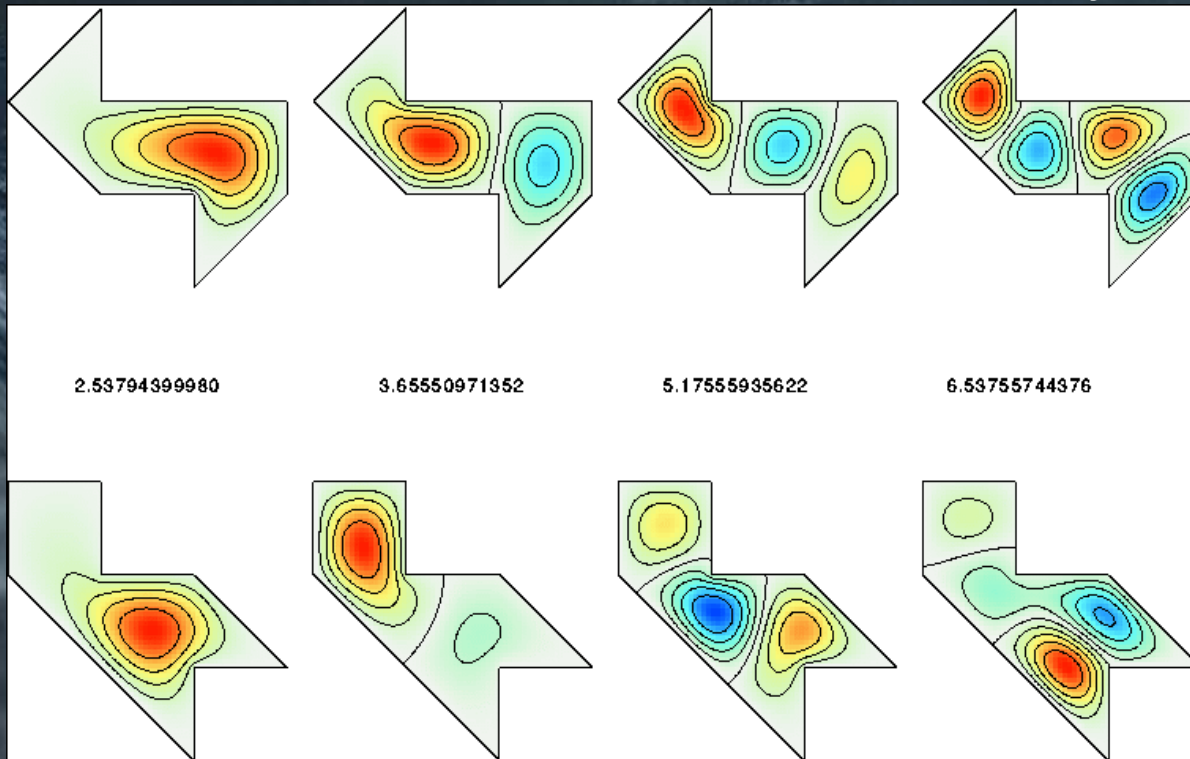
- Review of Problem Statement / History
- Motivation / Applications
- Methods Applied – Toy Problems
- Analysis
- Conclusions

Drumhead Problem

- Given a sample of a drums sound, attempt to calculate the amplitudes of the modes
 - time-series at location of microphone
- Related to a famous problem posed by mathematician Mark Kac (1966) asking: “Can One Hear the Shape of a Drum?”
- Lead to idea of iso-spectral drums

Isospectral Drums

- Drums with differing boundaries that have identical k_n values – so they sound alike!



Milnor, 1966 ...
Driscoll, 1997 SIAM

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Why Measure the Amplitudes?

- Being able to measure modes strengths suggests dynamics about the system.
- By measuring “windows” in time that overlap, the time-dependence of the amplitudes can be seen.
- Energy conservation – once a mode is excited, where does the energy go?
- Lead to prediction of amplitudes beyond the time window (forecasting).

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Methodology

- Three key issues:
 - Delta Function
 - Solution Architecture
 - Degeneracy of States
- Midn 1/C Grant Hundley
 - Developed a drum simulator
 - Working on numerical scheme to extract amplitudes from simulated time-series

Delta Function

$$u(x, t) = \sum_{n=0}^{\infty} [a_n(t) \sin(k_n x) + b_n(t) \cos(k_n x)]$$

$$a_m(t) = \frac{1}{L} \int_{-L}^{+L} u(x, t)_{data} \sin(k_m x) dx$$

$$a_m(t) = \frac{1}{L} \int_{-L}^{+L} \sum_{n=0}^{\infty} [a_n(t) \sin(k_n x) + b_n(t) \cos(k_n x)] \sin(k_m x) dx$$

$$a_m(t) = \sum_{n=0}^{\infty} \left[a_n(t) \frac{1}{L} \int_{-L}^{+L} \sin(k_n x) \sin(k_m x) dx + b_n(t) \frac{1}{L} \int_{-L}^{+L} \cos(k_n x) \sin(k_m x) dx \right]$$

$$a_m(t) = \sum_{n=0}^{\infty} [a_n(t) \delta_{nm} + b_n(t) \phi]$$

$$a_m(t) = \Delta_{mn} a_n(t)$$

$$a_n(t) = \oint u(x, t)_{data} \sin(k_n x) dx$$

$$b_n(t) = \oint u(x, t)_{data} \cos(k_n x) dx$$

Delta Function Assumptions

$$\omega_n = \frac{n*\pi}{2T} \text{ and } \omega_m = \frac{m*\pi}{2T}$$

ω_n and ω_m have integer relationship

$$(T_1 = 4T)$$

Wavelength is set from the window and is symmetric

$$\delta_{mn} = \frac{1}{T} \int_{-T}^{+T} \sin(\omega_n t) \sin(\omega_m t) dt$$

$$\emptyset = \frac{1}{T} \int_{-T}^{+T} \cos(\omega_n t) \sin(\omega_m t) dt$$

$$\Delta_{mn} = \frac{1}{T} \int_{-T}^{+T} \sin(\omega_n t) \sin(\omega_m t) dt$$

$$\varepsilon_{mn} = \frac{1}{T} \int_{-T}^{+T} \cos(\omega_n t) \sin(\omega_m t) dt$$

Define: Delta and Epsilon

Should these conditions fail.

Delta Function

$$\Delta_{mn} = \frac{1}{T} \int_{-T}^{+T} \sin(\omega_n t) \sin(\omega_m t) dt$$

$$\Delta_{mn} = \frac{1}{2T} \int_{-T}^{+T} [\cos((\omega_n - \omega_m)t) - \cos((\omega_n + \omega_m)t)] dt$$

$$\Delta_{mn} = \frac{1}{2T} \frac{1}{\omega_n - \omega_m} \sin((\omega_n - \omega_m)x) \Big|_{-T}^{+T} - \frac{1}{2T} \frac{1}{\omega_n + \omega_m} \sin((\omega_n + \omega_m)x) \Big|_{-T}^{+T}$$

$$\Delta_{mn} = \frac{1}{2T} \frac{1}{\frac{(n-m)\pi}{2T}} \sin\left(\frac{(n-m)\pi}{2T} x\right) \Big|_{-T}^{+T} - \frac{1}{2T} \frac{1}{\frac{(n+m)\pi}{2T}} \sin\left(\frac{(n+m)\pi}{2T} x\right) \Big|_{-T}^{+T}$$

$$\Delta_{mn} = \frac{2}{\pi(n-m)} \sin\left(\frac{\pi}{2}(n-m)\right) - \frac{2}{\pi(n+m)} \sin\left(\frac{\pi}{2}(n+m)\right), \quad \text{for even } n \rightarrow 2n$$

$$\Delta_{mn} = \text{sinc}(n-m) - \text{sinc}(n+m), \quad \text{assuming } m \text{ is free (real)}$$

$$\Delta_{\pm m, 2n} = \delta_{\pm m, 2n}, \quad \text{assuming } n \text{ is even and } m \text{ is an integer}$$

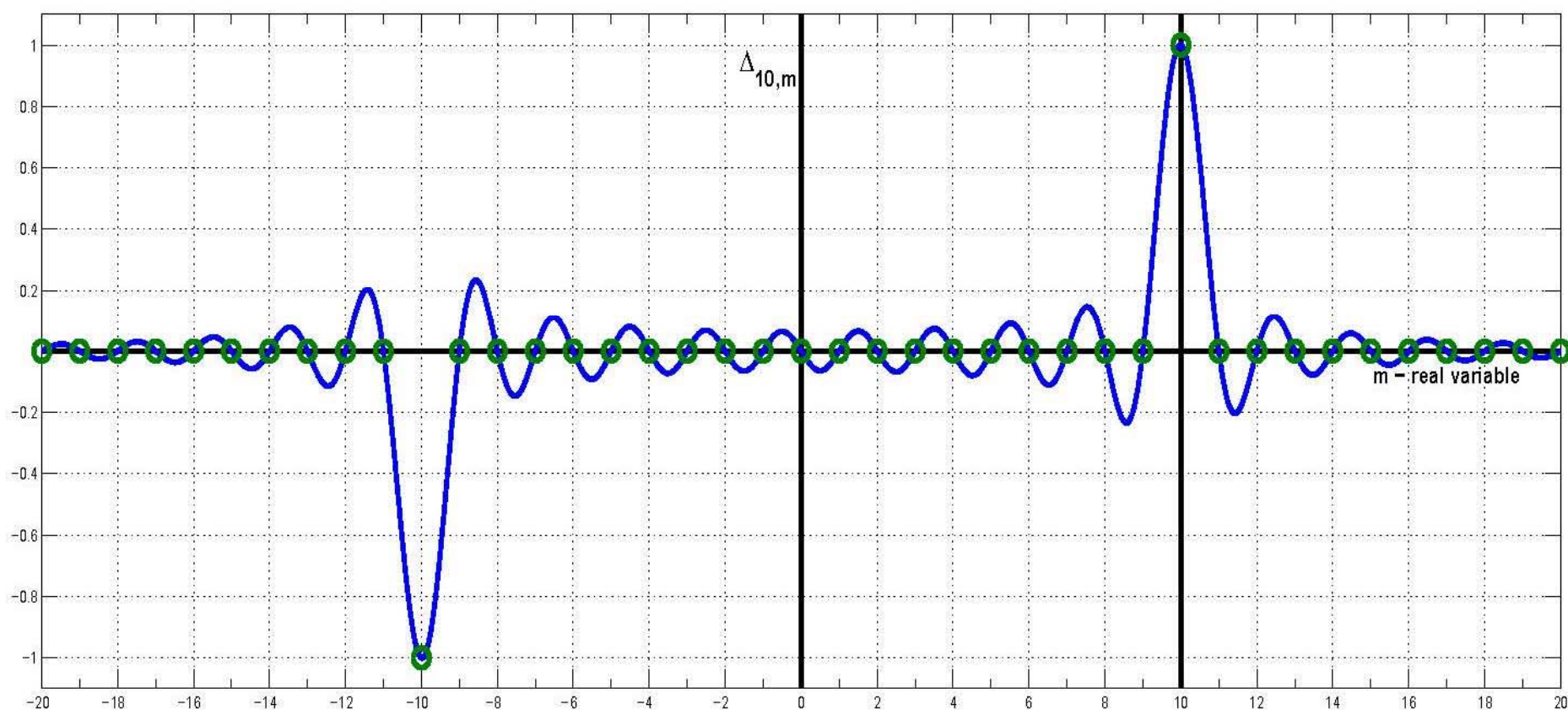
$$\varepsilon_{mn} = \frac{1}{T} \int_{-T}^{+T} \sin(\omega_n t) \cos(\omega_m t) dt$$

$$\varepsilon_{mn} = \frac{1}{2T} \int_{-T}^{+T} [\sin((\omega_n - \omega_m)t) - \sin((\omega_n + \omega_m)t)] dt$$

$$\varepsilon_{mn} = \phi$$

Delta Function $\sim \text{Sinc}(\pi \cdot (n-m))$

The Δ_{nm} function shown below shows its clear approximation to δ_{nm} when (n, m) are integers. Also shown are the twin responses at $(-m, +m)$ for the $n = 10$ case.



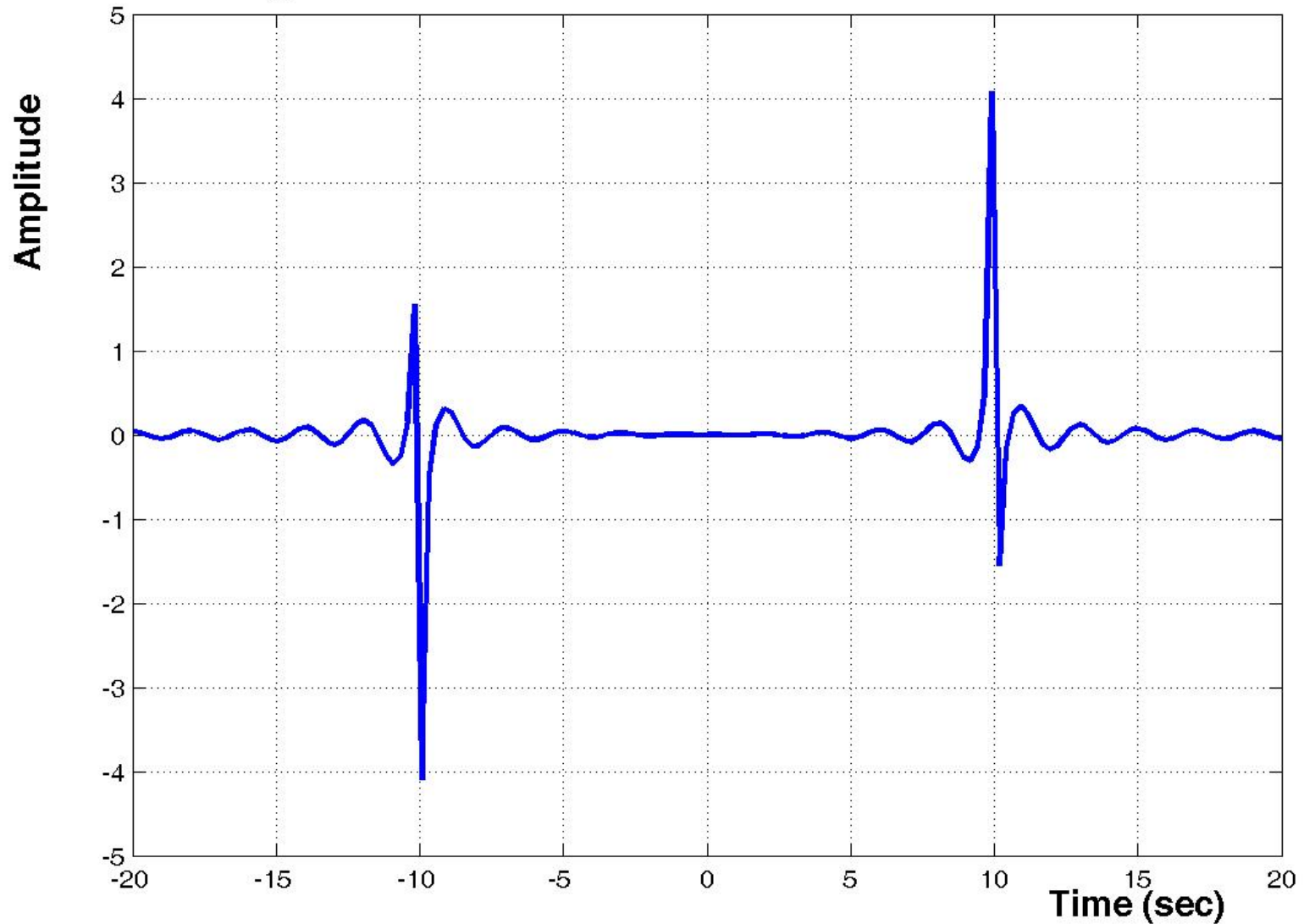
Delta & Epsilon Function

- Delta function is a measure of the “mixing” between the eigenmodes.
- Slightly different Delta function for the $\cos^* \cos$ term (more on this later).
- Epsilon will be identically zero for symmetric domains.

Epsilon(ω_m)

Epsilon Timing Resolution Function

$$\varepsilon(\omega_m=10) = \cos(\pi*(10-x))/(\pi*(10-x)) - \cos(\pi*(10+x))/(\pi*(10+x))$$



Solution Architecture: Solutions to space-time problems

$$u(x, t) = f(x) \cdot g(t)$$

$$f(x) = \sum_{n=0}^{\infty} A_n f_D(k_n x) + B_n f_N(k_n x)$$

$$g(t) = \sum_{n'=0}^{\infty} C_{n'} g_D(\omega_{n'} t) + D_{n'} g_N(\omega_{n'} t)$$

$$u(x, t) = \left[\sum_{n=0}^{\infty} A_n f_D(k_n x) + B_n f_N(k_n x) \right] \left[\sum_{n'=0}^{\infty} C_{n'} g_D(\omega_{n'} t) + D_{n'} g_N(\omega_{n'} t) \right]$$

$$u(x, t) = \left[\sum_{n=0}^{\infty} AC_n f_{D,n} g_{D,n} + BC_n f_{N,n} g_{D,n} + AD_n f_{D,n} g_{N,n} + BD_n f_{N,n} g_{N,n} \right]$$

$$u(x, t) = \left[\sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} AC_{nn'} f_{D,n} g_{D,n'} + BC_{nn'} f_{N,n} g_{D,n'} + AD_{nn'} f_{D,n} g_{N,n'} + BD_{nn'} f_{N,n} g_{N,n'} \right]$$

Nature of Eigenmodes with both space and time in solution

- Two types:
 - Coupled - for each k eigenvalue in space there exists a unique ω eigenvalue in time
 - Dispersion relationship, $\omega(k)$, for most differential equations in (x,t) .
 - Dispersion relationship is generally monotonic and increasing.
 - Decoupled – for systems not well motivated physically, yet can be described with a space-time basis set (river bank problem).

Solution Architecture

$$\alpha_m = \frac{1}{T} \int_{-T}^{+T} \sin(\omega_m t) u(x_1, t) dt$$

$$\beta_m = \frac{1}{T} \int_{-T}^{+T} \cos(\omega_m t) u(x_1, t) dt$$

At one location, x_1 , sample the data in a time-series.

Project out the sin and cosines.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix} = \frac{1}{T} \int_{-T}^{+T} \begin{pmatrix} g_D(\omega_1 t) \\ g_D(\omega_2 t) \\ g_D(\omega_3 t) \\ \vdots \\ g_D(\omega_m t) \\ g_N(\omega_1 t) \\ g_N(\omega_2 t) \\ g_N(\omega_3 t) \\ \vdots \\ g_N(\omega_m t) \end{pmatrix} \otimes \begin{pmatrix} g_D(\omega_1 t) & g_D(\omega_2 t) & \dots & g_D(\omega_n t) & \big| & g_N(\omega_1 t) & g_N(\omega_2 t) & \dots & g_N(\omega_n t) \end{pmatrix} dt \begin{pmatrix} f_D(k_1 x_1) \\ f_D(k_2 x_1) \\ & f_D(k_3 x_1) \\ & \ddots \\ & f_D(k_n x_1) & \big| & 0 \\ \hline 0 & \big| & f_D(k_1 x_1) \\ & f_D(k_2 x_1) \\ & f_D(k_3 x_1) \\ & \ddots \\ & f_D(k_n x_1) \end{pmatrix} \begin{pmatrix} AC_1 \\ AC_2 \\ AC_3 \\ \vdots \\ AC_n \\ AD_1 \\ AD_2 \\ AD_3 \\ \vdots \\ AD_n \end{pmatrix}$$

outer product becomes matrix of Δ_{mn} and ε_{mn}

Solution Architecture

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \dots & \Delta_{1n} & \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \dots & \varepsilon_{1n} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & & \Delta_{2n} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & & \varepsilon_{2n} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & & \Delta_{3n} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} & & \varepsilon_{3n} \\ \vdots & \vdots & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ \Delta_{m1} & \Delta_{m2} & \Delta_{m3} & \dots & \Delta_{mn} & \varepsilon_{m1} & \varepsilon_{m2} & \varepsilon_{m3} & \dots & \varepsilon_{mn} \end{pmatrix} \begin{pmatrix} f_D(k_1 x_1) \\ f_D(k_2 x_1) \\ f_D(k_3 x_1) \\ \vdots \\ f_D(k_n x_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} AC_1 \\ AC_2 \\ AC_3 \\ \vdots \\ AC_n \end{pmatrix}$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{21} & \varepsilon_{31} & \dots & \varepsilon_{n1} & \Delta_{11}^\circ & \Delta_{12}^\circ & \Delta_{13}^\circ & \dots & \Delta_{1n}^\circ \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{32} & & \varepsilon_{n2} & \Delta_{21}^\circ & \Delta_{22}^\circ & \Delta_{23}^\circ & & \Delta_{2n}^\circ \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} & & \varepsilon_{n3} & \Delta_{31}^\circ & \Delta_{32}^\circ & \Delta_{33}^\circ & & \Delta_{3n}^\circ \\ \vdots & \vdots & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ \varepsilon_{1m} & \varepsilon_{2m} & \varepsilon_{3m} & \dots & \varepsilon_{nm} & \Delta_{m1}^\circ & \Delta_{m2}^\circ & \Delta_{m3}^\circ & \dots & \Delta_{mn}^\circ \end{pmatrix} \begin{pmatrix} f_D(k_1 x_1) \\ f_D(k_2 x_1) \\ f_D(k_3 x_1) \\ \vdots \\ f_D(k_n x_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} AD_1 \\ AD_2 \\ AD_3 \\ \vdots \\ AD_n \end{pmatrix}$$

- Mixing nature of Delta, Epsilon matrix can clearly be seen.
- No nodes can exist in the $f(k \cdot x)$ matrix.
- Solve for Amplitudes thru matrix inversion.

Solution Architecture

- Short-hand notation:

$$\begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix} = \begin{pmatrix} \Delta_t & \varepsilon_t \\ \varepsilon_t^\dagger & \Delta_t^c \end{pmatrix} \begin{pmatrix} \vec{f}_D & 0 \\ 0 & \vec{f}_D \end{pmatrix} \begin{pmatrix} \vec{AC} \\ \vec{AD} \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \Delta_t & \varepsilon_t \\ \varepsilon_t^\dagger & \Delta_t^c \end{pmatrix} \begin{pmatrix} f_D & 0 \\ 0 & f_D \end{pmatrix} \begin{pmatrix} AC \\ AD \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{T} \int_{-T}^{+T} \begin{pmatrix} g_D \\ g_N \end{pmatrix} \otimes \begin{pmatrix} g_D & g_N \end{pmatrix} \begin{pmatrix} f_D & 0 \\ 0 & f_D \end{pmatrix} \begin{pmatrix} AC \\ AD \end{pmatrix} dt = \frac{1}{T} \int_{-T}^{+T} \begin{pmatrix} g_D \\ g_N \end{pmatrix} \otimes \begin{pmatrix} AC \cdot f_D \\ AD \cdot f_D \end{pmatrix} dt$$

$$\begin{pmatrix} \alpha_m \\ \beta_m \end{pmatrix} = \frac{1}{T} \int_{-T}^{+T} \begin{pmatrix} g_{Dm} \\ g_{Nm} \end{pmatrix} \otimes \sum_{n=0}^{\infty} \left[AC_n \cdot f_D(k_n \cdot x_1) \cdot g_{Dn} + AD_n \cdot f_D(k_n \cdot x_1) \cdot g_{Nn} \right] dt = \frac{1}{T} \int_{-T}^{+T} \begin{pmatrix} g_{Dm} \\ g_{Nm} \end{pmatrix} \otimes u(x_1, t) dt$$

Drum – Specific Solutions

- Drum problem: membrane stretched over a circular boundary (Dirichlet bc).
- Strike the drum.
- Use a microphone to record the time-series.
- Fourier analyze the time-series to obtain amplitudes for each ω_m : A_n

Drum

$$u(x, t) = \sum_{n=0}^{\infty} [A_n J_n(k_n r) \sin(n\theta) + B_n J_n(k_n r) \cos(n\theta)] [C_n \sin(\omega_n t) + D_n \cos(\omega_n t)]$$

$$u(r, t) = \sum_{(n, n')=0}^{\infty} [A_{nn'} J_n(k_{nn'} r) \sin(n\theta) + B_{nn'} J_n(k_{nn'} r) \cos(n\theta)] [C_{nn'} \sin(\omega_{nn'} t) + D_{nn'} \cos(\omega_{nn'} t)]$$

- Spatial modes are a combination of Bessel function, $J(k_n r)$ times $\sin(n\theta)$ or $\cos(n\theta)$
- Temporal modes use $\sin(\omega_n t)$ or $\cos(\omega_n t)$
- For each Bessel function, there exists multiple zero crossings, n'
- k_n values are non-integerlike, so ω_n fail conditions for orthonormality

$$\begin{pmatrix} f_{D1} \\ f_{D2} \\ f_{D3} \\ f_{D4} \\ f_{D5} \\ f_{D6} \\ f_{D7} \\ f_{D8} \\ f_{D9} \\ f_{D10} \\ f_{D11} \\ f_{D12} \\ f_{D13} \\ f_{D14} \\ f_{D15} \\ f_{D16} \\ f_{D17} \\ f_{D18} \\ f_{D19} \\ f_{D20} \\ \vdots \\ f_{Dn_{max}} \end{pmatrix} = \begin{pmatrix} J_0(k_{01} r_1) \sin(0 \theta_1) \\ J_0(k_{01} r_1) \cos(0 \theta_1) \\ J_1(k_{11} r_1) \sin(1 \theta_1) \\ J_1(k_{11} r_1) \cos(1 \theta_1) \\ J_2(k_{21} r_1) \sin(2 \theta_1) \\ J_2(k_{21} r_1) \cos(2 \theta_1) \\ J_0(k_{02} r_1) \sin(0 \theta_1) \\ J_0(k_{02} r_1) \cos(0 \theta_1) \\ J_3(k_{31} r_1) \sin(3 \theta_1) \\ J_3(k_{31} r_1) \cos(3 \theta_1) \\ J_1(k_{12} r_1) \sin(1 \theta_1) \\ J_1(k_{12} r_1) \cos(1 \theta_1) \\ J_4(k_{41} r_1) \sin(4 \theta_1) \\ J_4(k_{41} r_1) \cos(4 \theta_1) \\ J_2(k_{22} r_1) \sin(2 \theta_1) \\ J_2(k_{22} r_1) \cos(2 \theta_1) \\ J_0(k_{03} r_1) \sin(0 \theta_1) \\ J_0(k_{03} r_1) \cos(0 \theta_1) \\ J_3(k_{32} r_1) \sin(3 \theta_1) \\ J_3(k_{32} r_1) \cos(3 \theta_1) \\ \vdots \\ J_n(k_{nn'} r_1) \cos(n \theta_1) \end{pmatrix}$$

$$\begin{pmatrix} f_{D1} \\ f_{D2} \\ f_{D3} \\ \vdots \\ f_{Dn_{max}} \end{pmatrix} \begin{pmatrix} J_0(k_{01} r_1) \sin(0 \theta_1) \\ J_0(k_{01} r_1) \cos(0 \theta_1) \\ J_1(k_{11} r_1) \sin(1 \theta_1) \\ \vdots \\ J_n(k_{nn'} r_1) \cos(n \theta_1) \end{pmatrix}$$

Table 1: Bessel Function Zero Crossings

n	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Table 2: Bessel Function Zero Crossings

$k_{nn'}$	n	n'
2.4048	0	1
5.5201	0	2
8.6537	0	3
11.792	0	4
3.8317	1	1
7.0156	1	2
10.173	1	3
13.324	1	4
5.1356	2	1
8.4172	2	2
11.62	2	3
14.796	2	4
6.3802	3	1
9.761	3	2
13.015	3	3
16.223	3	4
7.5883	4	1
11.065	4	2
14.373	4	3
17.616	4	4

Table 3: Bessel Function Zero Crossings

$k_{nn'}$	n	n'
2.4048	0	1
3.8317	1	1
5.1356	2	1
5.5201	0	2
6.3802	3	1
7.0156	1	2
7.5883	4	1
8.4172	2	2
8.6537	0	3
9.761	3	2
10.173	1	3
11.065	4	2
11.62	2	3
11.792	0	4
13.015	3	3
13.324	1	4
14.373	4	3
14.796	2	4
16.223	3	4
17.616	4	4

Degeneracy of States

$$\underbrace{\left(g_D(\omega_1 t) \dots g_D(\omega_m t) \mid g_N(\omega_1 t) \dots g_N(\omega_m t) \right)}_{1 \times 2 \cdot m} \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & & 0 & 0 \\ \vdots & & & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 \end{pmatrix}}_{2 \cdot m \times 2 \cdot n} \underbrace{\begin{pmatrix} J_0(k_{01} r_1) \sin(0 \theta_1) \cdot AC_1 \\ J_0(k_{01} r_1) \cos(0 \theta_1) \cdot AC_2 \\ J_1(k_{11} r_1) \sin(1 \theta_1) \cdot AC_3 \\ J_1(k_{11} r_1) \cos(1 \theta_1) \cdot AC_4 \\ J_2(k_{21} r_1) \sin(2 \theta_1) \cdot AC_5 \\ \vdots \\ J_n(k_{nn'} r_1) \cos(n \theta_1) \cdot AC_n \end{pmatrix}}_{2 \cdot n \times 1} \\
 u(r, t) = \underbrace{\begin{pmatrix} J_0(k_{01} r_1) \sin(0 \theta_1) \cdot AD_1 \\ J_0(k_{01} r_1) \cos(0 \theta_1) \cdot AD_2 \\ J_1(k_{11} r_1) \sin(1 \theta_1) \cdot AD_3 \\ J_1(k_{11} r_1) \cos(1 \theta_1) \cdot AD_4 \\ J_2(k_{21} r_1) \sin(2 \theta_1) \cdot AD_5 \\ \vdots \\ J_n(k_{nn'} r_1) \cos(n \theta_1) \cdot AD_n \end{pmatrix}}_{2 \cdot n \times 1}$$

the Degeneracy Matrix $\equiv \mathbb{D}$

$$\underbrace{\left(g_D(\omega_1 t) \dots g_D(\omega_m t) \mid g_N(\omega_1 t) \dots g_N(\omega_m t) \right)}_{1 \times 2 \cdot m} \underbrace{\begin{pmatrix} J_0(k_{01} r_1) \sin(0 \theta_1) \cdot AC_1 + J_0(k_{01} r_1) \cos(0 \theta_1) \cdot AC_2 \\ J_1(k_{11} r_1) \sin(1 \theta_1) \cdot AC_3 + J_1(k_{11} r_1) \cos(1 \theta_1) \cdot AC_4 \\ J_2(k_{21} r_1) \sin(2 \theta_1) \cdot AC_5 + J_2(k_{21} r_1) \cos(2 \theta_1) \cdot AC_6 \\ \vdots \\ J_n(k_{nn'} r_1) \sin(n \theta_1) \cdot AC_n + J_n(k_{nn'} r_1) \cos(n \theta_1) \cdot AC_n \end{pmatrix}}_{2 \cdot n \times 1} \\
 u(r, t) = \underbrace{\begin{pmatrix} J_0(k_{01} r_1) \sin(0 \theta_1) \cdot AD_1 + J_0(k_{01} r_1) \cos(0 \theta_1) \cdot AD_2 \\ J_1(k_{11} r_1) \sin(1 \theta_1) \cdot AD_3 + J_1(k_{11} r_1) \cos(1 \theta_1) \cdot AD_4 \\ J_2(k_{21} r_1) \sin(2 \theta_1) \cdot AD_5 + J_2(k_{21} r_1) \cos(2 \theta_1) \cdot AD_6 \\ \vdots \\ J_n(k_{nn'} r_1) \sin(n \theta_1) \cdot AD_{n-1} + J_n(k_{nn'} r_1) \cos(n \theta_1) \cdot AD_n \end{pmatrix}}_{2 \cdot n \times 1}$$

Degeneracy and Sampling

- Due to symmetry in the solution, degenerate states are produced.
- Due to degeneracy, there is a 2-1 ratio of unknowns-knowns.
- Solution: add another sample location

$$\underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{pmatrix}}_{2 \cdot m \times 1} = \underbrace{\begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \dots & \Delta_{1m} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & & \Delta_{2m} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & & \Delta_{3m} \\ \vdots & & & \ddots & \vdots \\ \Delta_{m1} & \Delta_{m2} & \Delta_{m3} & \dots & \Delta_{mm} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \dots & \varepsilon_{1m} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & & \varepsilon_{2m} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} & & \varepsilon_{3m} \\ \vdots & & & \ddots & \vdots \\ \varepsilon_{m1} & \varepsilon_{m2} & \varepsilon_{m3} & \dots & \varepsilon_{mm} \end{pmatrix}}_{2 \cdot m \times 2 \cdot m} \underbrace{\begin{pmatrix} J_0(k_{01}r_1) \sin(0 \theta_1) \cdot AC_1 + J_0(k_{01}r_1) \cos(0 \theta_1) \cdot AC_2 \\ J_1(k_{11}r_1) \sin(1 \theta_1) \cdot AC_3 + J_1(k_{11}r_1) \cos(1 \theta_1) \cdot AC_4 \\ J_2(k_{21}r_1) \sin(2 \theta_1) \cdot AC_5 + J_2(k_{21}r_1) \cos(2 \theta_1) \cdot AC_6 \\ \vdots \\ J_n(k_{nn}r_1) \sin(n \theta_1) \cdot AC_{n-1} + J_n(k_{nn}r_1) \cos(n \theta_1) \cdot AC_n \end{pmatrix}}_{2 \cdot m \times 1}$$

$$\underbrace{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix}}_{2 \cdot m \times 1} = \underbrace{\begin{pmatrix} \varepsilon_{11} & \varepsilon_{21} & \varepsilon_{31} & \dots & \varepsilon_{m1} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{32} & & \varepsilon_{m2} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} & & \varepsilon_{m3} \\ \vdots & & & \ddots & \vdots \\ \varepsilon_{1m} & \varepsilon_{2m} & \varepsilon_{3m} & \dots & \varepsilon_{mm} \end{pmatrix} \begin{pmatrix} \Delta_{11}^\circ & \Delta_{12}^\circ & \Delta_{13}^\circ & \dots & \Delta_{1m}^\circ \\ \Delta_{21}^\circ & \Delta_{22}^\circ & \Delta_{23}^\circ & & \Delta_{2m}^\circ \\ \Delta_{31}^\circ & \Delta_{32}^\circ & \Delta_{33}^\circ & & \Delta_{3m}^\circ \\ \vdots & & & \ddots & \vdots \\ \Delta_{m1}^\circ & \Delta_{m2}^\circ & \Delta_{m3}^\circ & \dots & \Delta_{mm}^\circ \end{pmatrix}}_{2 \cdot m \times 2 \cdot m} \underbrace{\begin{pmatrix} J_0(k_{01}r_1) \sin(0 \theta_1) \cdot AD_1 + J_0(k_{01}r_1) \cos(0 \theta_1) \cdot AD_2 \\ J_1(k_{11}r_1) \sin(1 \theta_1) \cdot AD_3 + J_1(k_{11}r_1) \cos(1 \theta_1) \cdot AD_4 \\ J_2(k_{21}r_1) \sin(2 \theta_1) \cdot AD_5 + J_2(k_{21}r_1) \cos(2 \theta_1) \cdot AD_6 \\ \vdots \\ J_n(k_{nn}r_1) \sin(n \theta_1) \cdot AD_{n-1} + J_n(k_{nn}r_1) \cos(n \theta_1) \cdot AD_n \end{pmatrix}}_{2 \cdot m \times 1}$$

Second Location

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \\ \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}_{4 \cdot m \times 1} = \underbrace{\begin{pmatrix} \Delta_{11} & \Delta_{1m} & \varepsilon_{11} & \varepsilon_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{m1} & \Delta_{mm} & \varepsilon_{m1} & \varepsilon_{mm} \\ \varepsilon_{11} & \varepsilon_{m1} & \Delta_{11}^c & \Delta_{1m}^c \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon_{1m} & \varepsilon_{mm} & \Delta_{m1}^c & \Delta_{mm}^c \end{pmatrix}}_{\text{time-Delta matrix } 4 \cdot m \times 4 \cdot m} \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix}}_{\text{Degeneracy matrix } 4 \cdot m \times 4 \cdot n} \underbrace{\begin{pmatrix} J_0(k_{01} r_1) \sin(0 \theta_1) \\ J_0(k_{01} r_1) \cos(0 \theta_1) \\ J_1(k_{11} r_1) \sin(1 \theta_1) \\ J_1(k_{11} r_1) \cos(1 \theta_1) \\ J_2(k_{21} r_1) \sin(2 \theta_1) \\ \vdots \\ J_n(k_{nn'} r_1) \cos(n \theta_1) \end{pmatrix}}_{2 \cdot n \times 1} \underbrace{\begin{pmatrix} AC_1 \\ AC_2 \\ AC_3 \\ AC_4 \\ AC_5 \\ \vdots \\ AC_n \end{pmatrix}}_{2 \cdot n \times 1} \\
 + \underbrace{\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_m \\ \eta_1 \\ \vdots \\ \eta_m \end{pmatrix}}_{4 \cdot m \times 1} \underbrace{\begin{pmatrix} \Delta_{11} & \Delta_{1m} & \varepsilon_{11} & \varepsilon_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{m1} & \Delta_{mm} & \varepsilon_{m1} & \varepsilon_{mm} \\ \varepsilon_{11} & \varepsilon_{m1} & \Delta_{11}^c & \Delta_{1m}^c \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon_{1m} & \varepsilon_{mm} & \Delta_{m1}^c & \Delta_{mm}^c \end{pmatrix}}_{\text{time-Delta matrix } 4 \cdot m \times 4 \cdot m} \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix}}_{\text{Degeneracy matrix } 4 \cdot m \times 4 \cdot n} \underbrace{\begin{pmatrix} J_0(k_{01} r_2) \sin(0 \theta_2) \\ J_0(k_{01} r_2) \cos(0 \theta_2) \\ J_1(k_{11} r_2) \sin(1 \theta_2) \\ J_1(k_{11} r_2) \cos(1 \theta_2) \\ J_2(k_{21} r_2) \sin(2 \theta_2) \\ \vdots \\ J_n(k_{nn'} r_2) \cos(n \theta_2) \end{pmatrix}}_{2 \cdot n \times 1} \underbrace{\begin{pmatrix} AD_1 \\ AD_2 \\ AD_3 \\ AD_4 \\ AD_5 \\ \vdots \\ AD_n \end{pmatrix}}_{2 \cdot n \times 1} \\
 \text{The second sampled point} \Rightarrow \underbrace{\begin{pmatrix} J_0(k_{01} r_2) \sin(0 \theta_2) \\ J_0(k_{01} r_2) \cos(0 \theta_2) \\ J_1(k_{11} r_2) \sin(1 \theta_2) \\ J_1(k_{11} r_2) \cos(1 \theta_2) \\ J_2(k_{21} r_2) \sin(2 \theta_2) \\ \vdots \\ J_n(k_{nn'} r_2) \cos(n \theta_2) \end{pmatrix}}_{2 \cdot n \times 1} \underbrace{\begin{pmatrix} AC_1 \\ AC_2 \\ AC_3 \\ AC_4 \\ AC_5 \\ \vdots \\ AC_n \end{pmatrix}}_{2 \cdot n \times 1} + \underbrace{\begin{pmatrix} J_0(k_{01} r_2) \sin(0 \theta_2) \\ J_0(k_{01} r_2) \cos(0 \theta_2) \\ J_1(k_{11} r_2) \sin(1 \theta_2) \\ J_1(k_{11} r_2) \cos(1 \theta_2) \\ J_2(k_{21} r_2) \sin(2 \theta_2) \\ \vdots \\ J_n(k_{nn'} r_2) \cos(n \theta_2) \end{pmatrix}}_{2 \cdot n \times 1} \underbrace{\begin{pmatrix} AD_1 \\ AD_2 \\ AD_3 \\ AD_4 \\ AD_5 \\ \vdots \\ AD_n \end{pmatrix}}_{2 \cdot n \times 1} \\
 \text{Sample matrix } 4 \cdot n \times 2 \cdot n$$

Calculating the Amplitudes

$$\underbrace{\begin{pmatrix} AC_1 \\ AC_2 \\ AC_3 \\ AC_4 \\ AC_5 \\ \vdots \\ AC_n \end{pmatrix}}_{2 \cdot n \times 1} = \begin{pmatrix} \begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & & 0 & 0 \\ \vdots & & & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 \end{matrix} & \mathbf{O} \\ \mathbf{O} & \begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & & & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{matrix} \end{matrix} \right) \begin{pmatrix} J_0(k_{01} r_1) \sin(0 \theta_1) \\ J_0(k_{01} r_1) \cos(0 \theta_1) \\ J_1(k_{11} r_1) \sin(1 \theta_1) \\ J_1(k_{11} r_1) \cos(1 \theta_1) \\ J_2(k_{21} r_1) \sin(2 \theta_1) \\ \vdots \\ J_n(k_{nn'} r_1) \cos(n \theta_1) \end{pmatrix} \begin{matrix} \mathbf{O} \\ \mathbf{O} \\ \mathbf{O} \\ \mathbf{O} \\ \mathbf{O} \\ \mathbf{O} \\ \mathbf{O} \end{matrix} \begin{matrix} J_0(k_{01} r_1) \sin(0 \theta_1) \\ J_0(k_{01} r_1) \cos(0 \theta_1) \\ J_1(k_{11} r_1) \sin(1 \theta_1) \\ J_1(k_{11} r_1) \cos(1 \theta_1) \\ J_2(k_{21} r_1) \sin(2 \theta_1) \\ \vdots \\ J_n(k_{nn'} r_1) \cos(n \theta_1) \end{matrix} \end{pmatrix}^{-1} \underbrace{\begin{pmatrix} \Delta_t & \varepsilon_t & \mathbf{O} \\ \varepsilon_t^\dagger & \Delta_t^c & \mathbf{O} \\ \mathbf{O} & \Delta_t & \varepsilon_t \\ \mathbf{O} & \varepsilon_t^\dagger & \Delta_t^c \end{pmatrix}}_{4 \cdot m \times 4 \cdot m}^{-1} \underbrace{\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \\ \beta_1 \\ \vdots \\ \beta_m \\ \gamma_1 \\ \vdots \\ \gamma_m \\ \eta_1 \\ \vdots \\ \eta_m \end{pmatrix}}_{4 \cdot m \times 1}$$

$4 \cdot m \times 2 \cdot n$

Drum Solution in Short-hand

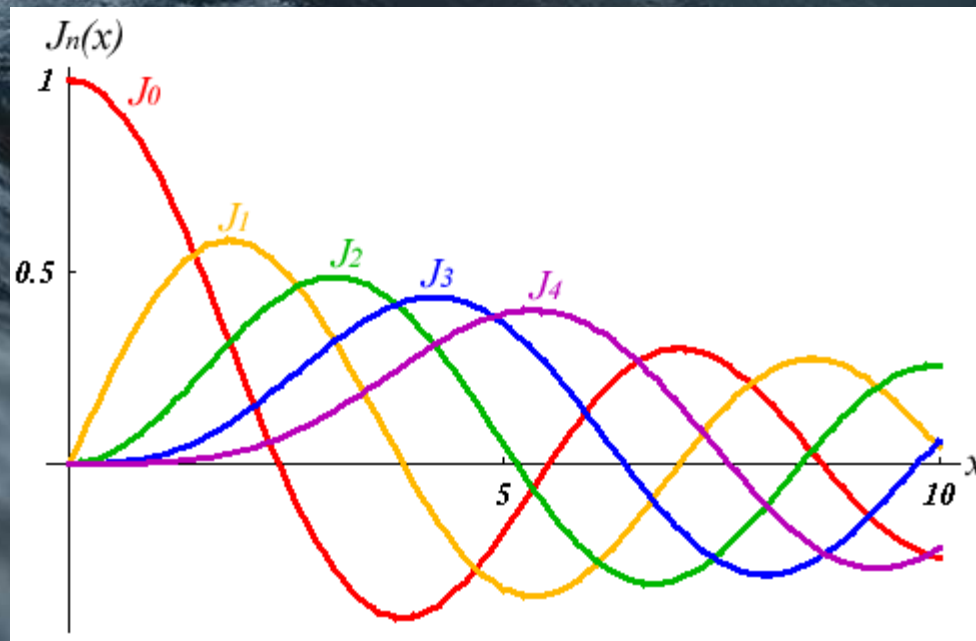
- Number of sample locations “squares-off” the degeneracy matrix, allowing the system to be solvable.

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \eta \end{pmatrix} = \begin{pmatrix} \Delta_t & 0 \\ 0 & \Delta_t \end{pmatrix} \left[\begin{pmatrix} \mathbb{D} & \end{pmatrix} \begin{pmatrix} f_{D1} & 0 \\ 0 & f_{D1} \\ f_{D2} & 0 \\ 0 & f_{D2} \end{pmatrix} \right] \begin{pmatrix} AC \\ AD \end{pmatrix}$$

$$\begin{pmatrix} AC \\ AD \end{pmatrix} = \left[\begin{pmatrix} \mathbb{D} & \end{pmatrix} \begin{pmatrix} f_{D1} & 0 \\ 0 & f_{D1} \\ f_{D2} & 0 \\ 0 & f_{D2} \end{pmatrix} \right]^{-1} \begin{pmatrix} \Delta_t & 0 \\ 0 & \Delta_t \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \eta \end{pmatrix}$$

Drum Problems

- $n=0$ $\sin(n\theta)$ term needs to be removed, breaking the 2-1 ratio to less than 2-1.
- Test method against drum simulation, with known inputs to the amplitudes, A_n .
- Further degeneracies exist due to closeness of k -eigenvalues.



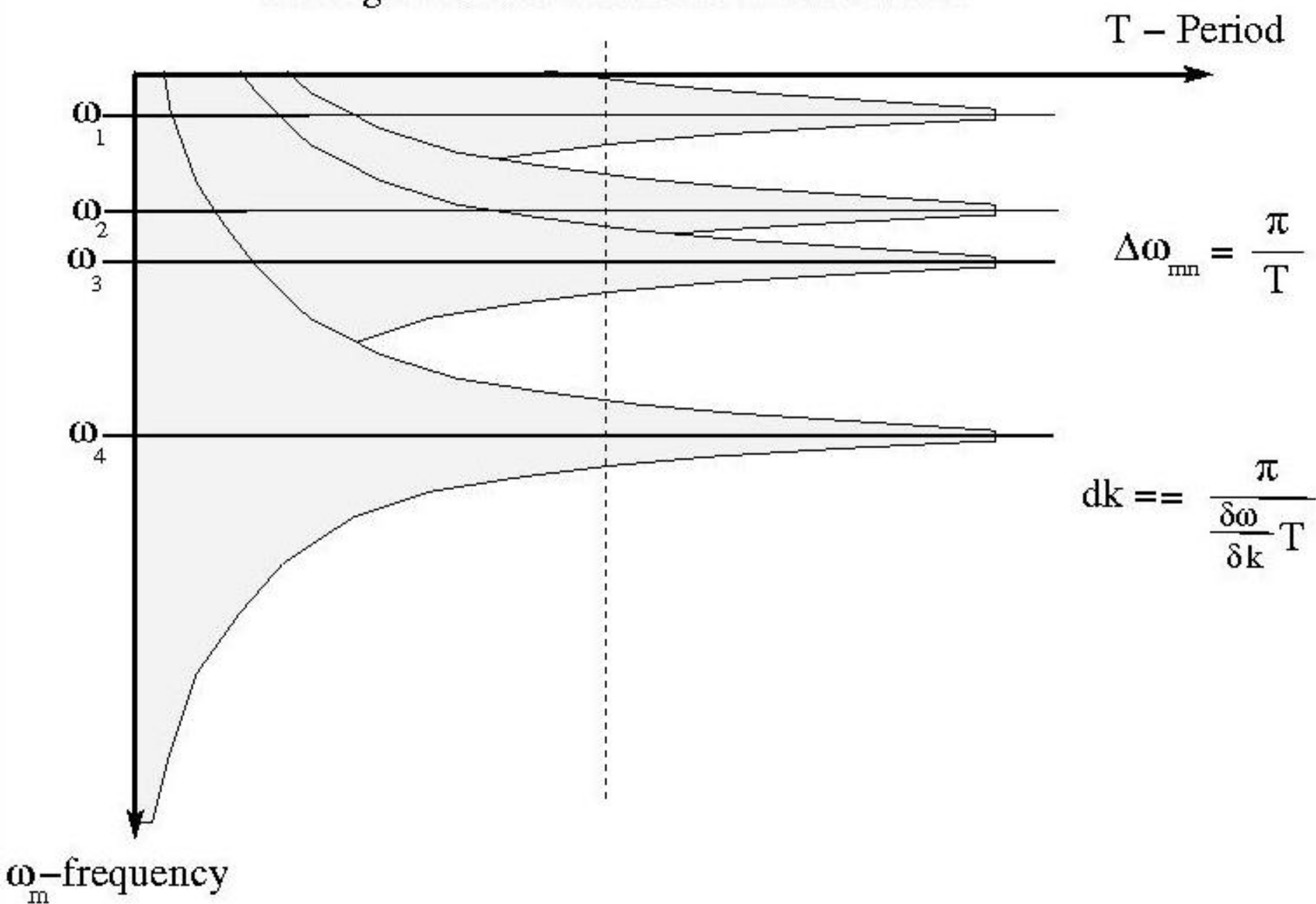
Chesapeake Bay Problems

- No known analytic solution to the Bay
(ie. No analytic hints as to any degeneracy)
- At a given samples location, calculate the projections, (α_m, β_m) for a range of (ω_m, T) .
- Observe the patterns of ω_m and compare the sequence to k_n 's.
- Guess the dispersion relationship (map from k_n to ω_m)

Degeneracy of States

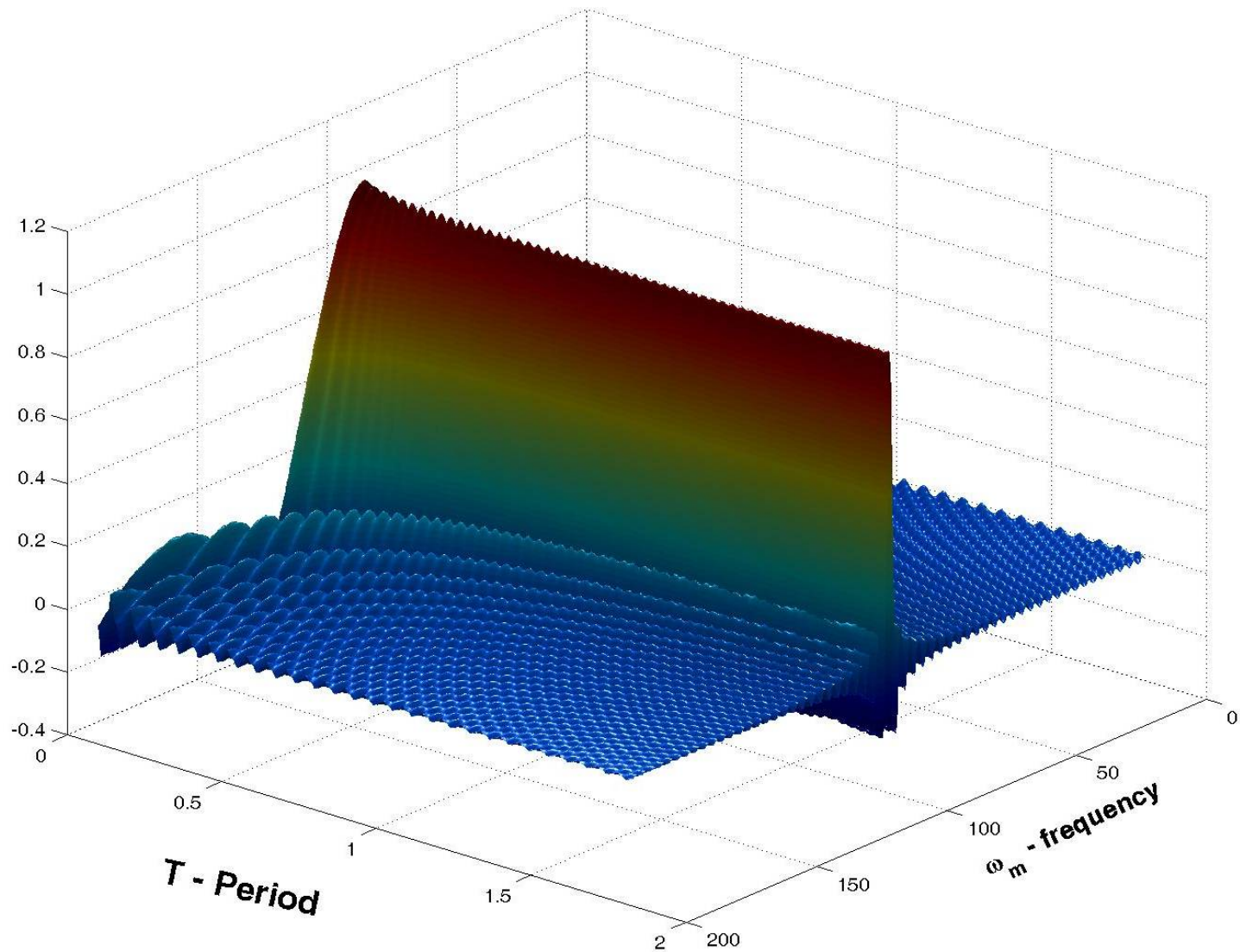
- From observables at frequencies ω_m , observe projection changes as the period, T , is changed.
- Select an appropriate period, T .
- Construct Degeneracy matrix based on best guess of dispersion relation as well as k -eigenvalues density.

Timing Resolution Nature of Delta Matrix



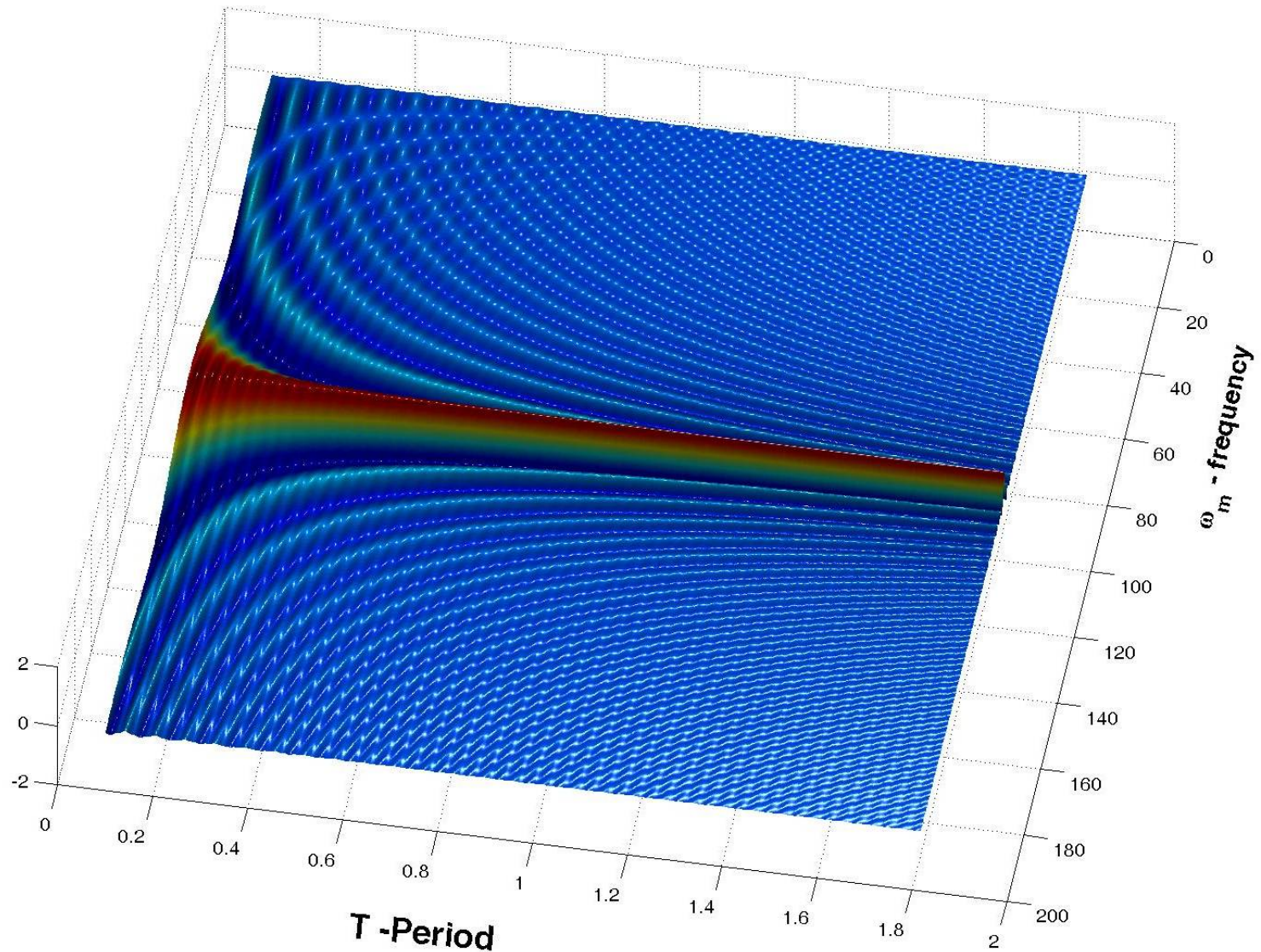
Delta(ω_m, T)

Timing Resolution - Delta Matrix - $\Delta(T, \omega_m)$



Delta(ω_m, T)

Timing Resolution - Delta Matrix - $\Delta(T, \omega_m)$



Future Plans:

- Run thru the toy model (Drum).
- Add source terms to Chesapeake Bay model.
- Add Delta(spatial) matrix.

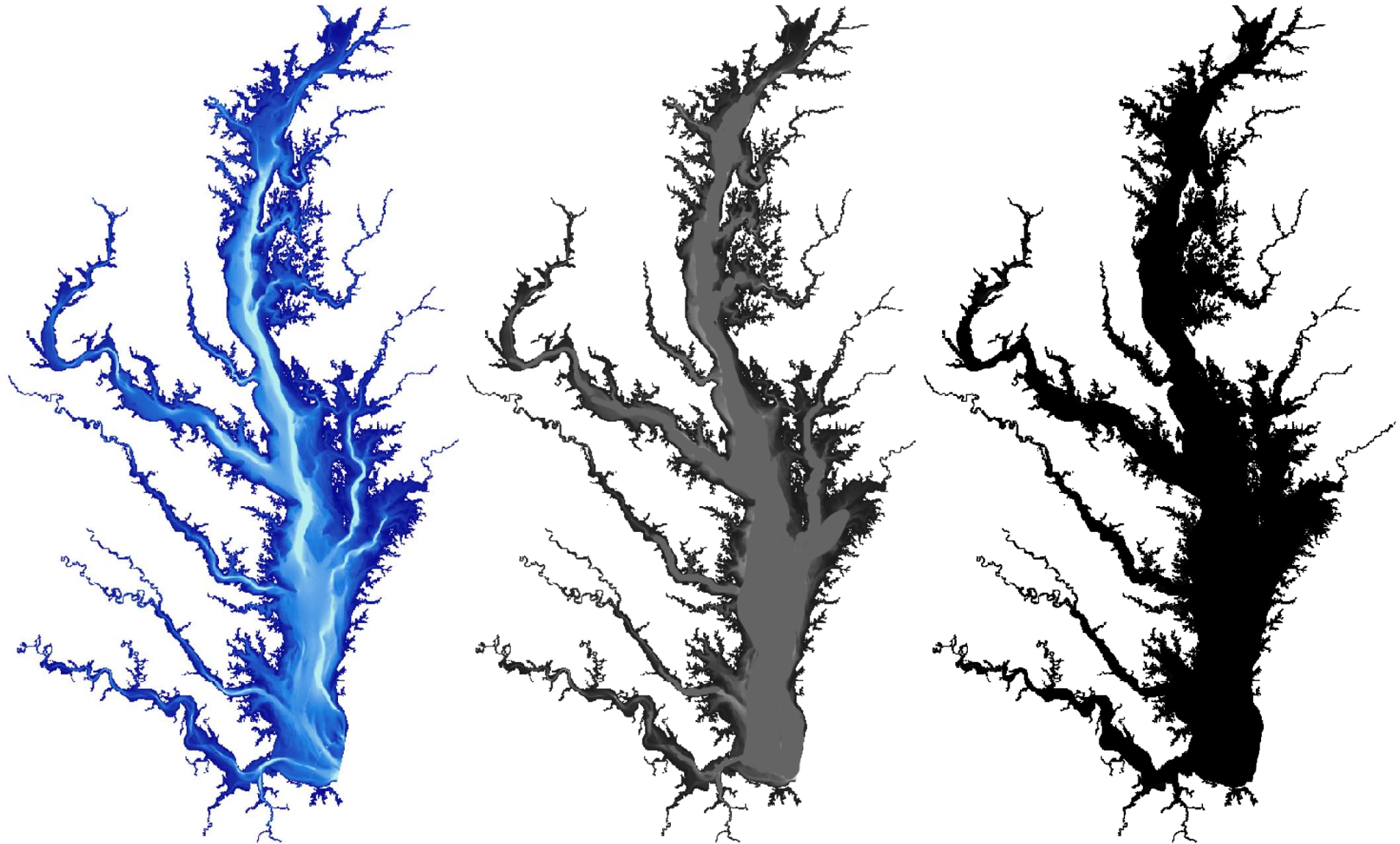
Acknowledgements

- Student: MIDN 1/C Grant Hundley
- Collaborator: Reza Malek-Madani
- James W. Kinnear (USN – Ret.)

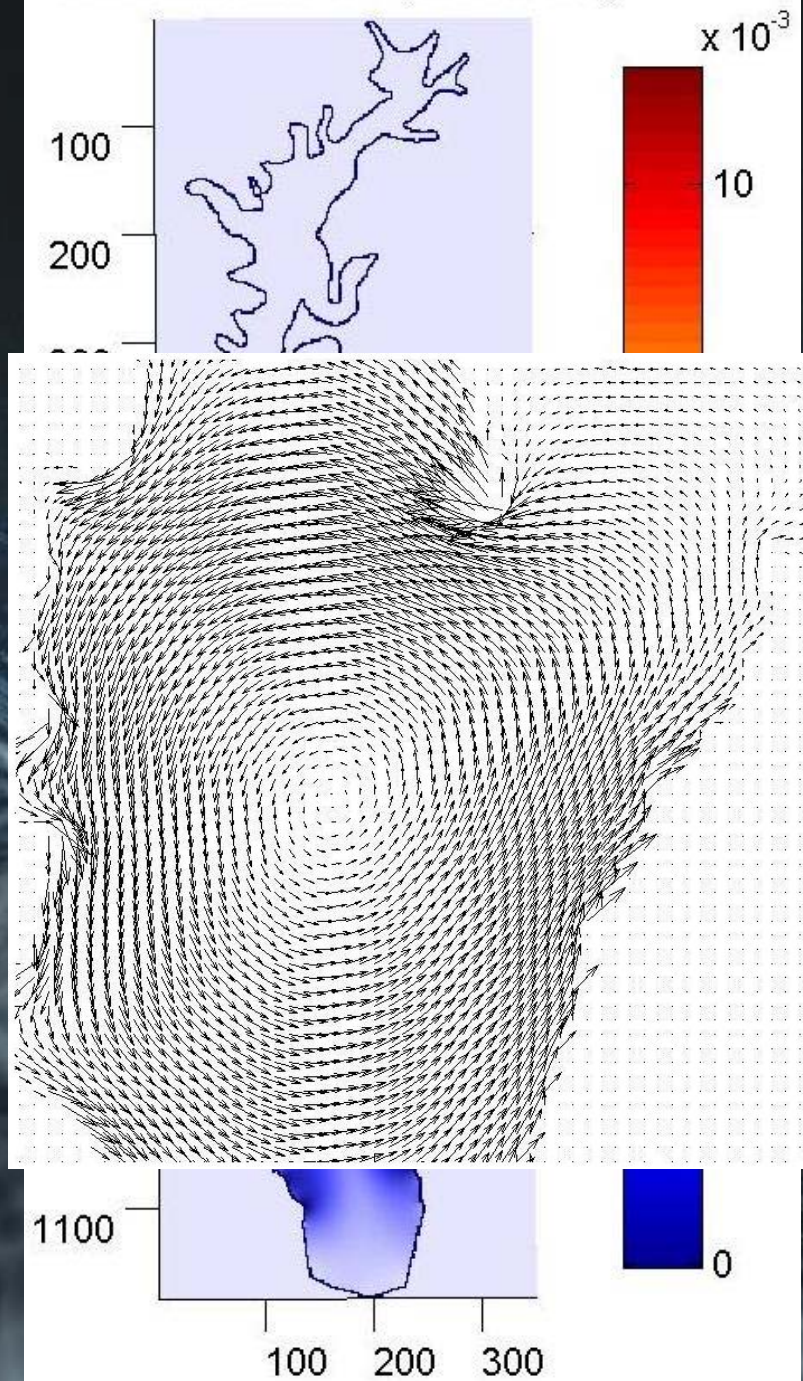
Chesapeake Bay Problem

- Take data at stations around the Bay, collecting time-series of vector flows.
- How many stations are needed to provide enough data to fully calculate the modes?

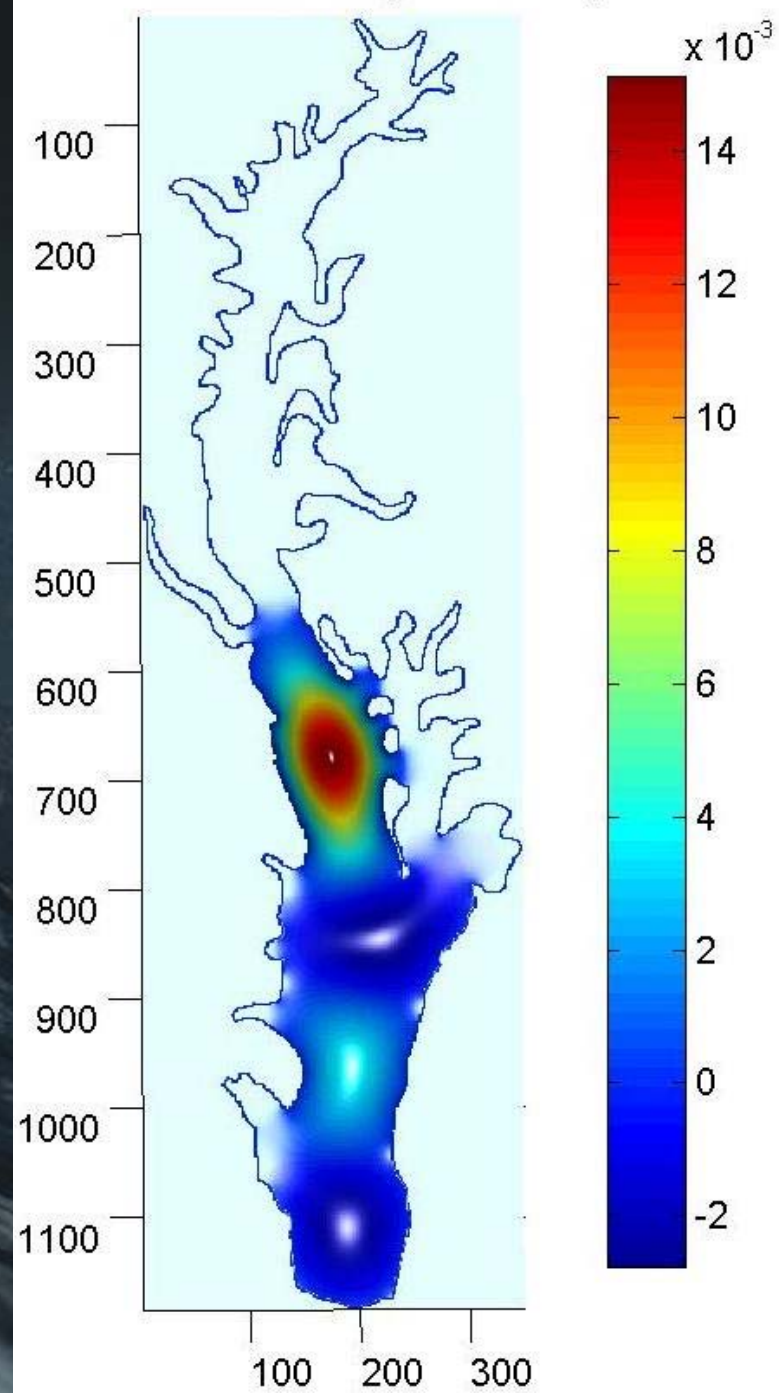
Image Processing of the Chesapeake



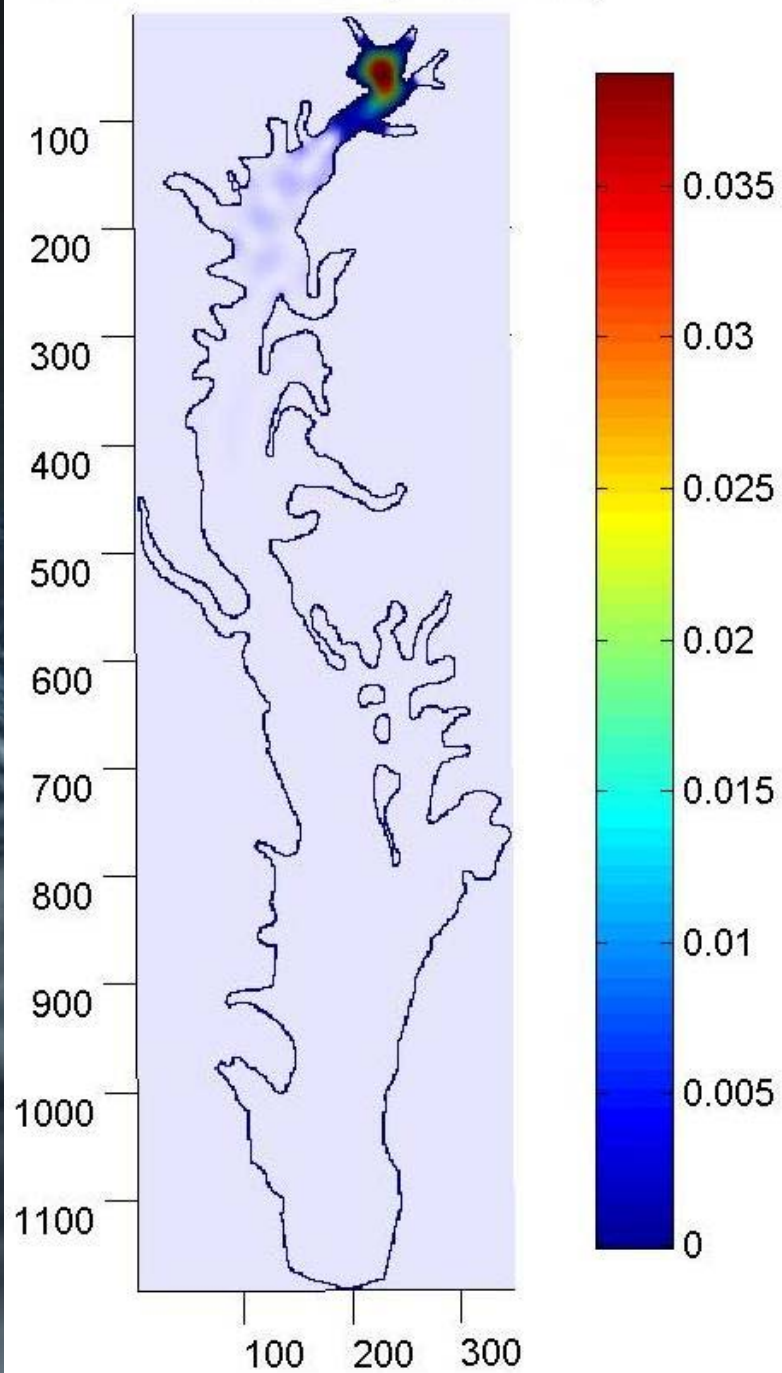
Dirichlet Mode 1 (350X1185)



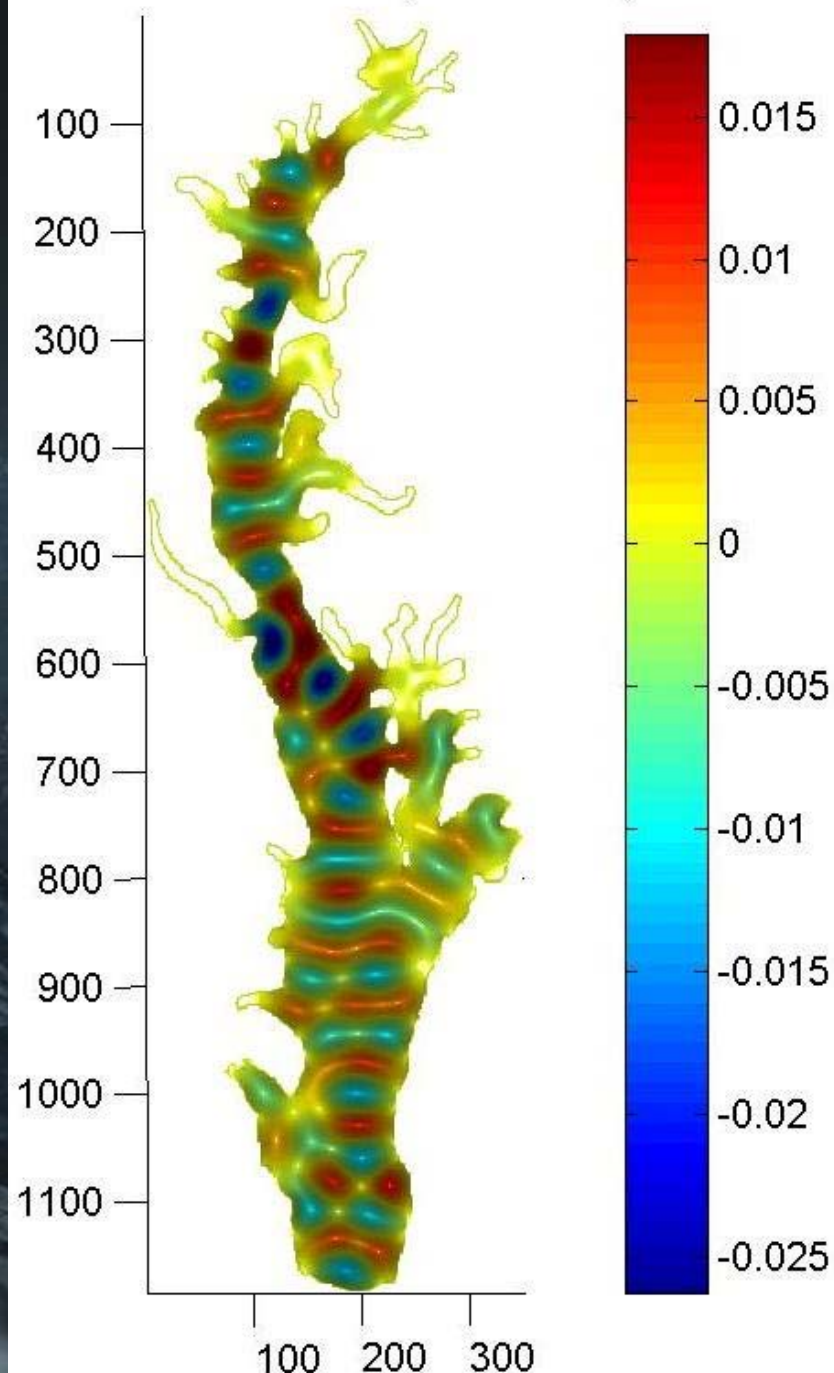
Dirichlet Mode 4 (350X1185)



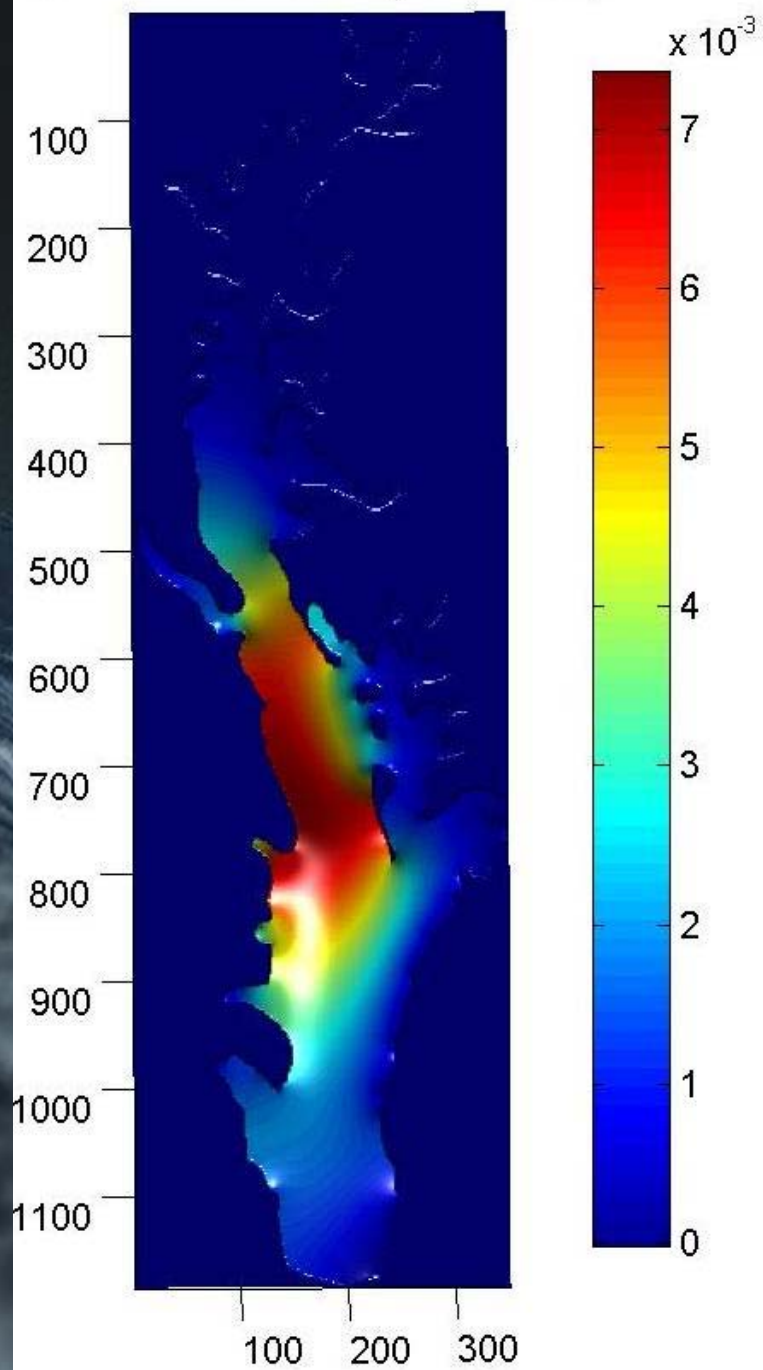
Dirichlet Mode 52 (350X1185)



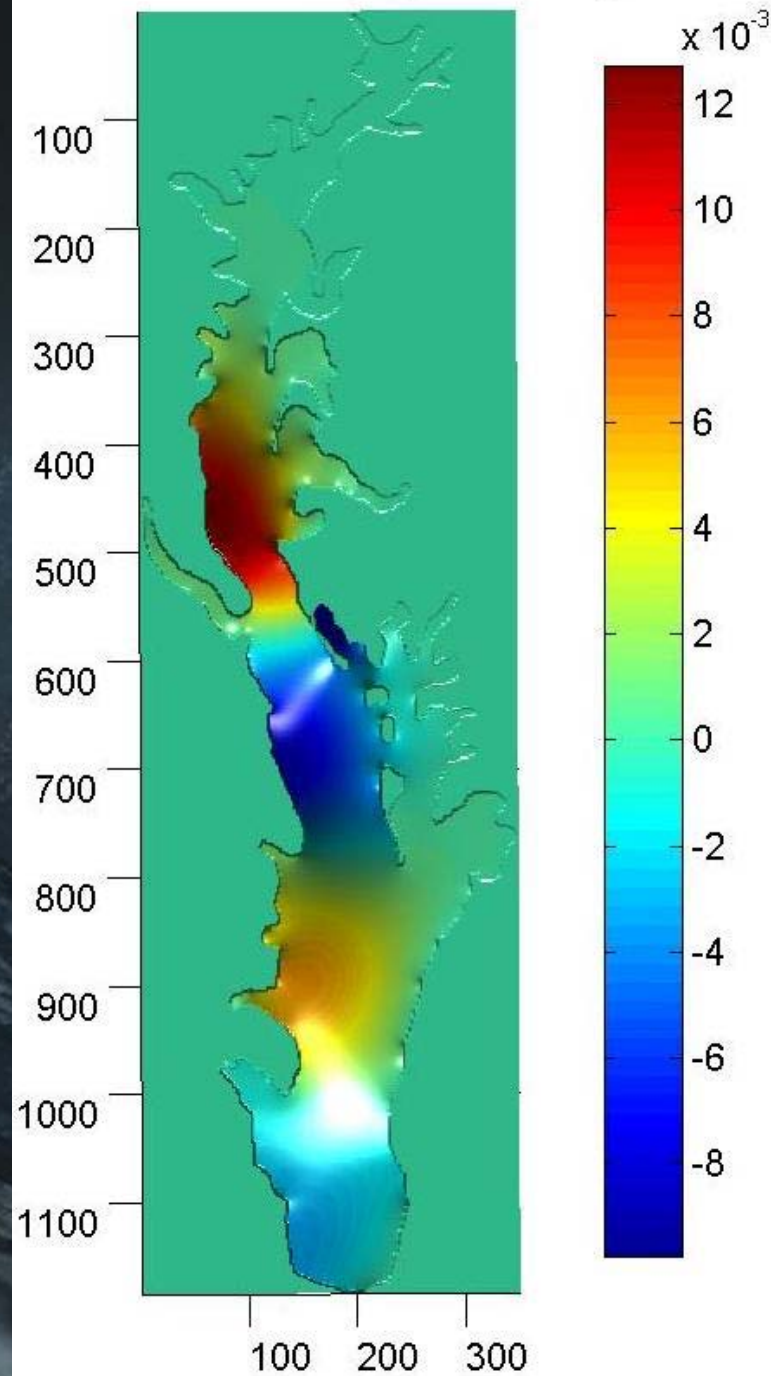
Dirichlet Mode 98 (350X1185)



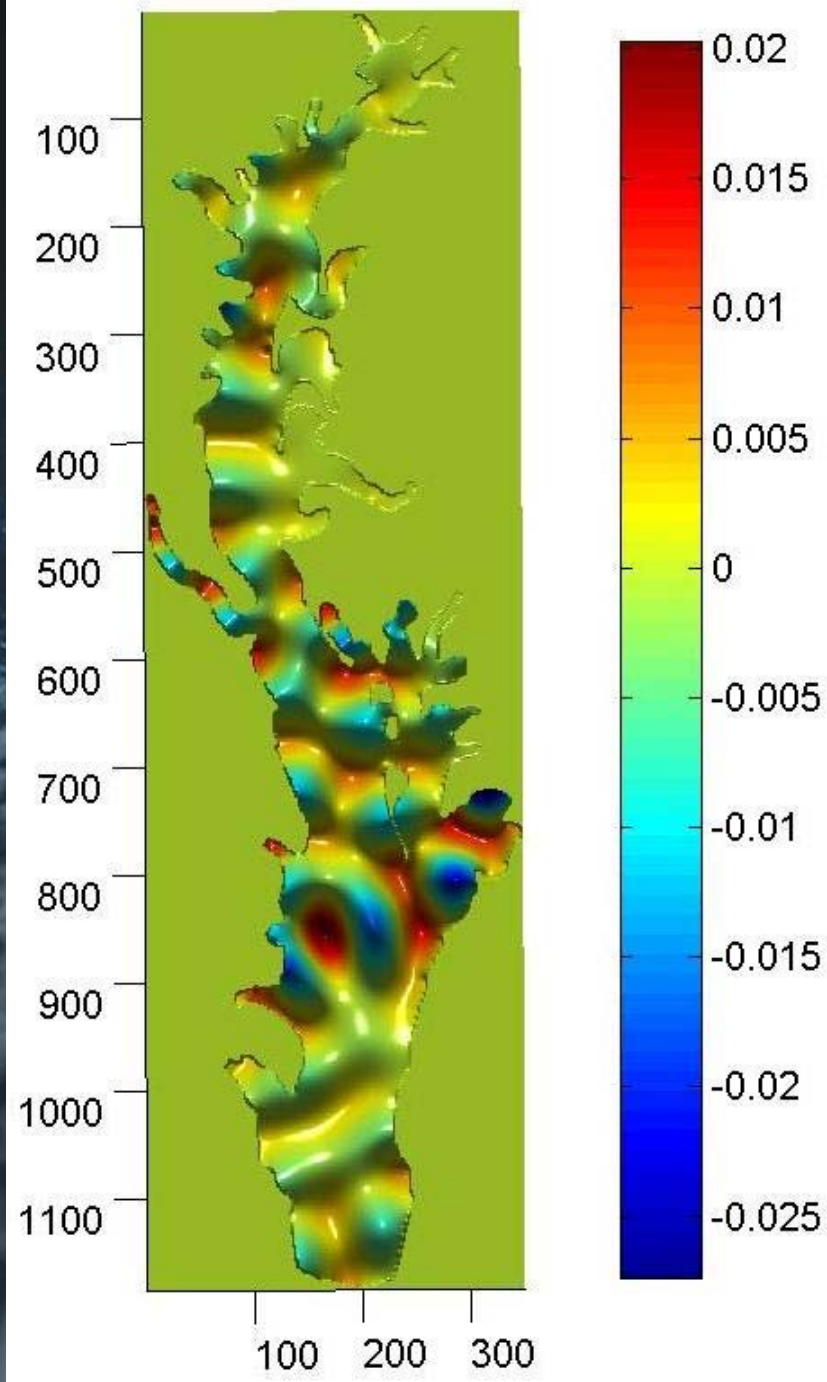
Neumann Mode 1 (350X1185)



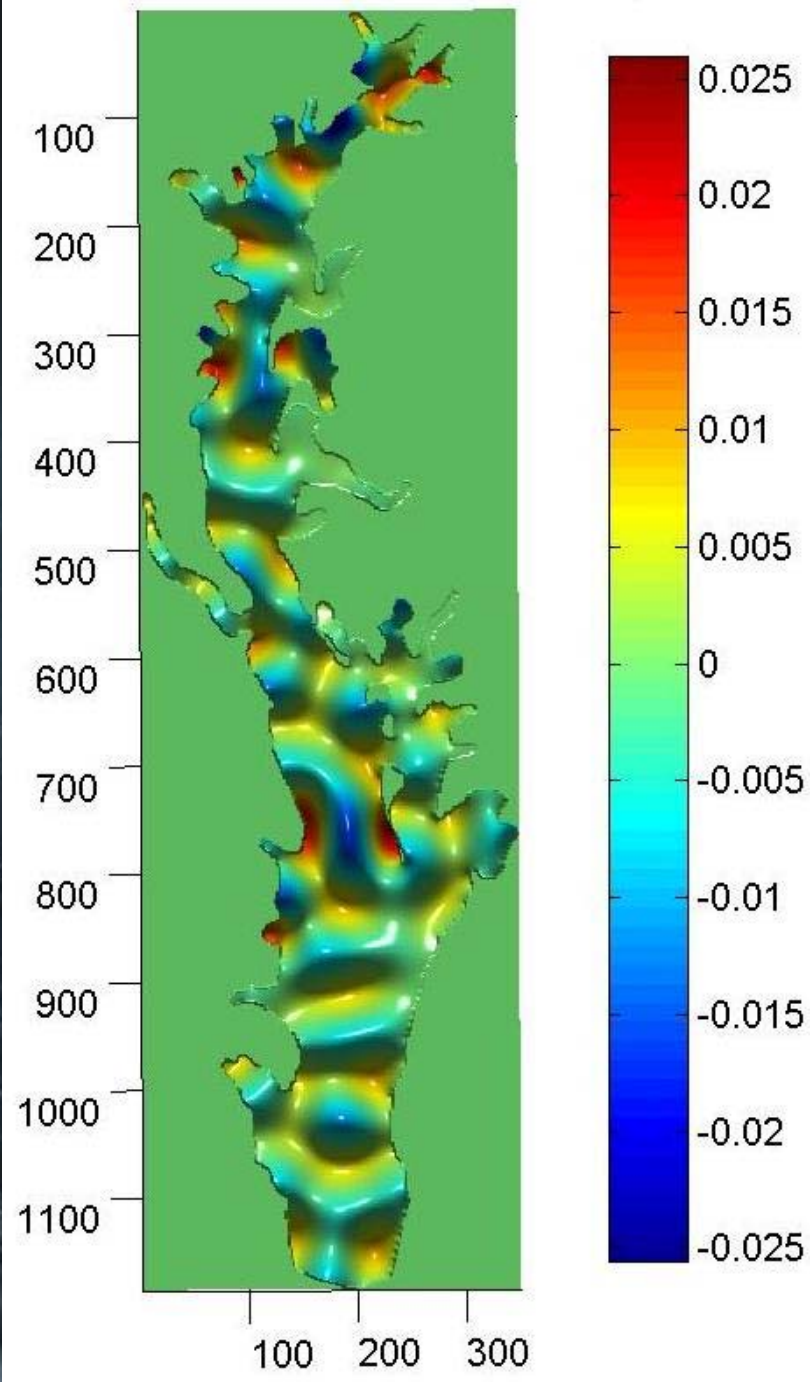
Neumann Mode 4 (350X1185)



Neumann Mode 91 (350X1185)



Neumann Mode 100 (350X1185)



Advantages

- Eigenmodes fill domain (space), suggest future behavior

